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Rotor Blade Stability in Turbulent Flows—Part II

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As a sequel to Part I, this paper is directed toward the effect of turbulence in the atmosphere on the rotor blade stability during a forward flight. Two types of motion are considered: uncoupled flapping, and coupled flapping and torsion. Assuming that the turbulence velocities which appear as parametric excitations can be approximated by white noise processes, the method of the Markov process is applied in the formulation of the problem. The results are presented as stability boundaries for the first- and second-order stochastic moments corresponding to different spectral levels of the excitations. The stability boundaries for the nonturbulence case, previously obtained from deterministic analyses, are also included for comparison. It is shown that the uncoupled flapping motion remains quite stable under the turbulence intensities and operating conditions, which may be reasonably expected in the service life of a helicopter, but the stability region for the coupled flapping and torsional motion is significantly reduced due to normal turbulence.

Nomenclature

a	= lift curve slope
$\bar{B}$	= matrix appearing in the first-moment equation
$\bar{\bar{B}}$	= matrix appearing in the second-moment equation
c	= blade chord
$E[\ ]$	= ensemble average
F	$=(I_\beta/16I_\alpha)\ (c/R)^2$
I_α	= feathering mass moment of inertia of a blade, kg-m^2
I_β	= flapping mass moment of inertia of a blade, kg-m^2
m_j	= drift coefficients in Itô's stochastic differential equation
p	= blade flapping frequency/blade angular velocity
Q	$=cI_\beta/(4RI_\alpha)$
$\bar{Q}$	= Floquet transition matrix
R	= rotor radius
t	= time
U	= lateral turbulence component, m/s
v	= forward flying velocity, m/s
V	= longitudinal turbulence component, m/s
α	= torsion angle
β	= flapping angle
γ	$=R^4\rho ca/I_\beta$, blade Lock number
ϵ	= a small parameter
η	$=V/\Omega R$
μ	$=v/\Omega R$, advance ratio
ξ	$=U/\Omega R$
ρ	= air density
σ_{ij}	= diffusion coefficients in Itô's stochastic differential equation
σ_v^2	= mean-square value of random velocity V
$\Phi_{\eta\eta},\Phi_{\xi\xi}$	= spectral densities of nondimensional velocities η and ξ with respect to nondimensional frequency
$\Phi_{\eta\xi},\Phi_{\xi\eta}$	= cross-spectral densities of nondimensional velocities η and ξ with respect to nondimensional frequency
$\bar{\Phi}_{\eta\eta},\bar{\Phi}_{\xi\xi}$	= spectral densities of nondimensional velocities η and ξ with respect to physical frequency, rad/s
$\bar{\bar{\Phi}}_{VV}$	= spectral densities of physical velocity $V(t)$ with respect to physical frequency
ψ	$=\Omega t$, azimuth angle (nondimensional time)
Ω	= blade angular velocity, rad/s
ω_c	= cutoff frequency
ω_α	= torsion frequency/Ω
$(\dot{\ })$	= derivative with respect to azimuth angle ψ

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Introduction

THE stability of helicopter rotor blades in coupled flapping and torsional motions and operating in an idealized smooth flow was investigated by Sissingh and Kuczynski.[1] Since strong turbulent flows are often encountered by a helicopter in its service life, the question arises as to whether their presence will induce instability of an otherwise stable motion. To answer this question, the equations of motion were rederived to include the random turbulence terms.[2] It was found that for flapping and torsional motions only the horizontal turbulence components appeared in the coefficients on the left-hand sides, while both horizontal and vertical components appeared in the nonhomogeneous terms on the right-hand sides. Since the inhomogeneous terms do not affect the system stability, they may be dropped in the stability considerations. The final version of the equations, with inhomogeneous and higher order turbulence terms removed, may be cast in a matrix form:

$$\begin{Bmatrix}\ddot{\beta}\\ \ddot{\alpha}\end{Bmatrix}+(\gamma/2)\begin{bmatrix}\bar{C} & 0\\ 6Q\bar{\ell}_{r\dot{\beta}} & 6F\bar{C}_\alpha\end{bmatrix}\begin{Bmatrix}\dot{\beta}\\ \dot{\alpha}\end{Bmatrix}+\begin{bmatrix}p^2+(\gamma/2)\bar{K} & -(\gamma/2)\bar{m}_\alpha\\ 3\gamma Q\bar{\ell}_{r\beta} & \omega_\alpha^2+3\gamma Q\bar{K}_\alpha\end{bmatrix}\begin{Bmatrix}\beta\\ \alpha\end{Bmatrix}=0 \tag{1}$$

where coefficients $\bar{C}$, $\bar{K}$,... are related to the nondimensional horizontal velocities $\xi(t)$ and $\eta(t)$ as follows:

$$\begin{Bmatrix}\bar{C}\\ \bar{K}\\ \bar{m}_\alpha\\ \bar{C}_\alpha\\ \bar{K}_\alpha\\ \bar{\ell}_{r\beta}\\ \bar{\ell}_{r\dot{\beta}}\end{Bmatrix}=\begin{bmatrix}C & C_\xi & C_\eta\\ K & K_\xi & K_\eta\\ m_\alpha & m_{\alpha\xi} & m_{\alpha\eta}\\ C_\alpha & C_{\alpha\xi} & C_{\alpha\eta}\\ K_\alpha & K_{\alpha\xi} & K_{\alpha\eta}\\ \ell_{r\beta} & \ell_{r\beta\xi} & \ell_{r\beta\eta}\\ \ell_{r\dot{\beta}} & \ell_{r\dot{\beta}\xi} & \ell_{r\dot{\beta}\eta}\end{bmatrix}\begin{Bmatrix}1\\ \xi\\ \eta\end{Bmatrix} \tag{2}$$

The elements of the 7×3 matrix in Eq. (2) are periodic functions of the azimuth angle ψ.

If the correlation times of the excitations $\xi(t)$ and $\eta(t)$ are short compared with the relaxation time of the response, then the response is close to a Markov stochastic process governed by an equivalent Itô stochastic equation. This equivalent equation can be obtained by a stochastic averaging method.[3,4]

In the reduced case of uncoupled flapping motion in a hovering flight, closed-form solutions have been obtained for the transition probability density of the response amplitude and the stability condition for the stochastic moment of an arbitrary order in Ref. 2. However, the real threat of turbulence to the stability of a rotor blade appears to be in the coupled motions and in high forward-speed flights. These will be considered in this paper.

High Forward-Speed Flights

When the advance ratio μ is not small, the periodic variation in the coefficients $\bar{C}, \bar{K},\ldots$ can no longer be ignored. This suggests that, although the concept of stochastic average of Stratonovich is still sound for high forward-speed flights, the time-average portion of the whole procedure requires further consideration. One guideline for any stochastic analysis is that the results should be reducible to those of the corresponding deterministic analysis when the random terms are set to zero. The deterministic results available in the literature are those due to Sissingh,[5] and Peters and Hohenemser[6] for the uncoupled flapping motion, and those due to Sissingh and Kuczynski[1] for coupled flapping torsional motion, all of which are based on unaveraged equations. In order to compare with these results, the time-average portion of the Stratonovich method cannot be used. Instead, use will be made of the Wong and Zakai corrections to obtain the drift and diffusion coefficients for the equivalent Itô equation.[7] (See also Appendix A of Ref. 2.) This means that the random excitations $\xi(t)$ and $\eta(t)$ are replaced by "physical" white noise processes and time averaging becomes unnecessary.

A. Uncoupled Flapping Motion

Setting $\alpha=0$ in the first row of Eq. (1), one obtains the equation for uncoupled flapping motion:

$$\ddot{\beta}+(\gamma/2)\bar{C}\dot{\beta}+[p^2+(\gamma/2)\bar{K}]\beta=0 \tag{3}$$

Let $X_1=\beta$ and $X_2=\dot{\beta}$. Equation (3) is replaced by two equations of the standard form [Eq. (A13) of Ref. 2]:

$$\mathrm{d}X_j/\mathrm{d}\psi=\epsilon f_j(X,\psi)+\epsilon^{1/2}g_{jk}(X,\psi)\xi_k(\psi) \tag{4}$$

where

$$\epsilon f_1=X_2,\quad \epsilon f_2=-(p^2+(\gamma/2)K)X_1-(\gamma/2)CX_2$$

$$\epsilon^{1/2}g_{11}=\epsilon^{1/2}g_{12}=0,\quad \epsilon^{1/2}g_{21}=-(\gamma/2)(K_\xi X_1+C_\xi X_2)$$

$$\epsilon^{1/2}g_{22}=-(\gamma/2)(K_\eta X_1+C_\eta X_2),\quad \xi_1(\psi)=\xi(t),$$

$$\xi_2(\psi)=\eta(t) \tag{5}$$

The drift terms can then be obtained by an application of the Wong and Zakai corrections:

$$\epsilon \boldsymbol{m}=\bar{\boldsymbol{B}}X \tag{6}$$

where

$$\bar{\boldsymbol{B}}=\begin{bmatrix} 0 & 1 \\ -p^2-(\gamma/2)K+C_{21} & -(\gamma/2)C+C_{22}\end{bmatrix} \tag{7}$$

$$C_{21}=\pi\left(\frac{\gamma}{2}\right)^2[C_\xi K_\xi\Phi_{\xi\xi}+(C_\xi K_\eta\Phi_{\xi\eta}+C_\eta K_\xi\Phi_{\eta\xi})+C_\eta K_\eta\Phi_{\eta\eta}]$$

$$C_{22}=\pi\left(\frac{\gamma}{2}\right)^2[C_\xi^2\Phi_{\xi\xi}+C_\eta C_\xi(\Phi_{\xi\eta}+\Phi_{\eta\xi})+C_\eta^2\Phi_{\eta\eta}] \tag{8}$$

The elements of the product diffusion matrix $\epsilon(\boldsymbol{\sigma\sigma}')$ are all zero, except

$$\begin{aligned}\epsilon(\boldsymbol{\sigma\sigma}')_{22}&=(2\pi)\epsilon[g_{21}^2\Phi_{\xi\xi}+g_{21}g_{22}(\Phi_{\xi\eta}+\Phi_{\eta\xi})+g_{22}^2\Phi_{\eta\eta}]\\&=S_{31}X_1^2+S_{32}X_1X_2+S_{33}X_2^2\end{aligned} \tag{9}$$

where

$$S_{31}=\pi\frac{\gamma^2}{2}[K_\xi^2\Phi_{\xi\xi}+K_\eta K_\xi(\Phi_{\xi\eta}+\Phi_{\eta\xi})+K_\eta^2\Phi_{\eta\eta}]$$

$$S_{32}=\pi\frac{\gamma^2}{2}[2K_\xi C_\xi+(K_\xi C_\eta+K_\eta C_\xi)(\Phi_{\xi\eta}+\Phi_{\eta\xi})+2K_\eta C_\eta]$$

$$S_{33}=2C_{22} \tag{10}$$

It follows that the elements of the diffusion matrix $\epsilon^{1/2}\boldsymbol{\sigma}$ also are all zero, except

$$\epsilon^{1/2}\sigma_{22}=[S_{31}X_1^2+S_{32}X_1X_2+S_{33}X_2^2]^{1/2}$$

Thus, the Itô equation for X is completely determined:

$$\mathrm{d}X_j=\bar{B}_{jk}X_k\mathrm{d}\psi+[S_{31}X_1^2+S_{32}X_1X_2+S_{33}X_2^2]^{1/2}\delta_{2j}\mathrm{d}W_2 \tag{11}$$

where δ_{2j} is a Kronecker delta and W_2 is a Wiener process.

The equations for the first moments of X_j are obtained by taking the ensemble average of Eq. (11), resulting in

$$\mathrm{d}E[X_j]/\mathrm{d}\psi=\bar{B}_{jk}E[X_k] \tag{12}$$

Since the elements of matrix $\bar{\boldsymbol{B}}$ are complicated periodic functions of ψ, a closed-form solution is not obtainable. However, Eq. (12) is now deterministic, and it has the same form as the one solved by Peters and Hohenemser[6] using a numerical procedure. Briefly, this technique involves the numerical integration of Eq. (12) to obtain a transfer relationship between $E[X(0)]$ and $E[X(2\pi)]$ as follows:

$$E[X(2\pi)]=\bar{Q}E[X(0)] \tag{13}$$

where $\bar{Q}$ is the so-called Floquet transition matrix. The eigenvalues of $\bar{Q}$ are generally complex; they can be expressed as exponentials $\Lambda_k=\exp[2\pi(\lambda_k+iw_k)]$. The blade response is stable if all the λ_k are negative, and unstable when one or more of the λ_k become positive.

To determine the stability boundary for the second moments, one applies Itô's differential rule (Appendix A, Ref. 2) to obtain three stochastic equations for $Y_1=X_1^2$, $Y_2=X_1X_2$, and $Y_3=X_2^2$, and then takes the ensemble averages. The results can be cast in a matrix form:

$$\mathrm{d}E[Y]/\mathrm{d}\psi=\bar{\bar{\boldsymbol{B}}}E[Y] \tag{14}$$

where

$$[\bar{\bar{\boldsymbol{B}}}]=\begin{bmatrix}0 & 2 & 0\\ \bar{B}_{21} & \bar{B}_{22} & 1\\ S_{31} & 2\bar{B}_{21}+S_{32} & 2\bar{B}_{22}+S_{33}\end{bmatrix} \tag{15}$$

and $\bar{B}_{jk}$ is the (j,k) element of matrix $\bar{\boldsymbol{B}}$. The stability condition of Eq. (15) can be investigated again using the numerical method of the Floquet transition matrix.

B. Coupled Flapping—Torsional Motion

In principle, the problem is the same as that of the uncoupled flapping motion, but algebraically it is much more tedious. Therefore, only the final results will be presented herein. Let $X=\{\beta,\dot{\beta},\alpha,\dot{\alpha}\}$. The first objective is to evaluate matrices $\bar{B}$ and $\bar{\bar{B}}$ analogous to those in Eqs. (12) and (14), since these matrices are intrinsically related to their respective Floquet transition matrices. It can be shown that[8]

$$\bar{B}=\begin{bmatrix} 0 & 1 & 0 & 0 \\ D_{21}+C_{21} & D_{22}+C_{22} & D_{23}+C_{23} & 0 \\ 0 & 0 & 0 & 1 \\ D_{41}+C_{41} & D_{42}+C_{42} & D_{43}+C_{43} & D_{44}+C_{44} \end{bmatrix} \tag{16}$$

where

$$D_{21}=-p^2-(\gamma/2)K \qquad C_{21}=T_{2221}$$

$$D_{22}=-(\gamma/2)C \qquad C_{22}=T_{2232}$$

$$D_{23}=(\gamma/2)m_\alpha \qquad C_{23}=T_{2322}$$

$$D_{41}=-3\gamma Q\ell_{r\beta} \qquad C_{41}=T_{4221}+T_{4441}$$

$$D_{42}=-3\gamma Q\ell_{r\dot{\beta}} \qquad C_{42}=T_{4222}+T_{4442}$$

$$D_{43}=-\omega_\alpha^2-3\gamma QK_\alpha \qquad C_{43}=T_{4223}+T_{4443}$$

$$D_{44}=-3\gamma FC_\alpha$$

$$T_{2221}=\pi(\gamma/2)^2[K_\eta C_\eta\Phi_{\eta\eta}+K_\xi C_\xi\Phi_{\xi\xi}+(C_\eta K_\xi+C_\xi K_\eta)\Phi_{\eta\xi}]$$

$$T_{2222}=\pi(\gamma/2)^2[C_\eta^2\Phi_{\eta\eta}+C_\xi^2\Phi_{\xi\xi}+2C_\eta C_\xi\Phi_{\eta\xi}]$$

$$T_{2322}=-\pi(\gamma/2)^2[C_\eta m_{\alpha\eta}\Phi_{\eta\eta} + C_\xi m_{\alpha\xi}\Phi_{\xi\xi}+(C_\eta m_{\alpha\xi}+C_\xi m_{\alpha\eta})\Phi_{\eta\xi}]$$

$$T_{4221}=(3/2)\pi\gamma^2 Q[K_\eta\ell_{r\dot{\beta}\eta}\Phi_{\eta\eta}+K_\xi\ell_{r\dot{\beta}\xi}\Phi_{\xi\xi} + (K_\eta\ell_{r\dot{\beta}\xi}+K_\xi\ell_{r\dot{\beta}\eta})\Phi_{\eta\xi}]$$

$$T_{4222}=(3/2)\pi\gamma^2 Q[\ell_{r\dot{\beta}\eta}C_\eta\Phi_{\eta\eta}+\ell_{r\dot{\beta}\xi}C_\xi\Phi_{\xi\xi} + (\ell_{r\dot{\beta}\eta}C_\xi+\ell_{r\dot{\beta}\xi}C_\eta)\Phi_{\eta\xi}]$$

$$T_{4223}=-(3/2)\pi\gamma^2 Q[\ell_{r\dot{\beta}\eta}m_{\alpha\eta}\Phi_{\eta\eta} +\ell_{r\dot{\beta}\xi}m_{\alpha\xi}\Phi_{\xi\xi}+(\ell_{r\dot{\beta}\eta}m_{\alpha\xi}+\ell_{r\dot{\beta}\xi}m_{\alpha\eta})\Phi_{\eta\xi}]$$

$$T_{4441}=9\pi\gamma^2 FQ[\ell_{r\beta\eta}C_{\alpha\eta}\Phi_{\eta\eta} +\ell_{r\beta\xi}C_{\alpha\xi}\Phi_{\xi\xi}+(\ell_{r\beta\eta}C_{\alpha\xi}+\ell_{r\beta\xi}C_{\alpha\eta})\Phi_{\eta\xi}]$$

$$T_{4442}=9\pi\gamma^2 FQ[\ell_{r\dot{\beta}\eta}C_{\alpha\eta}\Phi_{\eta\eta} +\ell_{r\dot{\beta}\xi}C_{\alpha\xi}\Phi_{\xi\xi}+(\ell_{r\dot{\beta}\eta}C_{\alpha\xi}+\ell_{r\dot{\beta}\xi}C_{\alpha\eta})\Phi_{\eta\xi}]$$

$$T_{4443}=9\pi\gamma^2 FQ[K_{\alpha\eta}C_{\alpha\eta}\Phi_{\eta\eta} +K_{\alpha\xi}C_{\alpha\xi}\Phi_{\xi\xi}+(K_{\alpha\eta}C_{\alpha\xi}+K_{\alpha\xi}C_{\alpha\eta})\Phi_{\eta\xi}]$$

$$T_{4444}=9\pi\gamma^2 F^2[C_{\alpha\eta}^2\Phi_{\eta\eta}+C_{\alpha\xi}^2\Phi_{\xi\xi}+2C_{\alpha\eta}C_{\alpha\xi}\Phi_{\eta\xi}] \tag{17}$$

In the preceding expressions, it has been assumed that $\Phi_{\xi\eta}=\Phi_{\eta\xi}$. This assumption implies that both cross-spectra are purely real, which is reasonable since $\xi(\psi)$ and $\eta(\psi)$ are idealized as physical white noise processes.

Matrix $\bar{\bar{S}}$ is 10×10, corresponding to the ten second moments $E[X_1^2]$, $E[X_1X_2]$, $E[X_1X_3]$, $E[X_1X_4]$, $E[X_2^2]$, $E[X_2X_3]$, $E[X_2X_4]$, $E[X_3^2]$, $E[X_3X_4]$, and $E[X_4^2]$. Specifically

$$\bar{\bar{S}}=\begin{bmatrix} 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \bar{B}_{21} & \bar{B}_{22} & \bar{B}_{23} & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ \bar{B}_{41} & \bar{B}_{42} & \bar{B}_{43} & \bar{B}_{44} & 0 & 0 & 1 & 0 & 0 & 0 \\ S_{51} & S_{52}+2\bar{B}_{21} & S_{53} & 0 & S_{55}+2\bar{B}_{23} & S_{56}+2\bar{B}_{23} & 0 & S_{58} & 0 & 0 \\ 0 & 0 & \bar{B}_{21} & 0 & 0 & \bar{B}_{22} & 1 & \bar{B}_{23} & 0 & 0 \\ S_{71} & S_{72}+\bar{B}_{41} & S_{73} & S_{74}+\bar{B}_{21} & S_{75}+\bar{B}_{42} & S_{76}+\bar{B}_{43} & S_{77}+\bar{B}_{22}+\bar{B}_{44} & S_{78} & S_{79}+\bar{B}_{23} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & \bar{B}_{41} & 0 & 0 & \bar{B}_{42} & 0 & \bar{B}_{43} & \bar{B}_{44} & 1 \\ S_{10,1} & S_{10,2} & S_{10,3} & S_{10,4}+2\bar{B}_{41} & S_{10,5} & S_{10,6} & S_{10,7}+2\bar{B}_{42} & S_{10,8} & S_{10,9}-2\bar{B}_{43} & S_{10,10}+2\bar{B}_{44} \end{bmatrix} \tag{18}$$

where

$$S_{51}=V_{2121} \qquad S_{52}=4T_{2221} \qquad S_{53}=2V_{2321} \qquad S_{55}=2T_{2222} \qquad S_{56}=4T_{2322}$$

$$S_{58}=V_{2323} \qquad S_{71}=V_{4121} \qquad S_{72}=2T_{4221}+T_{4122} \qquad S_{73}=V_{4321}+V_{4123} \qquad S_{74}=V_{4421}$$

$$S_{75}=2T_{4222} \qquad S_{76}=V_{4322}+2T_{4223} \qquad S_{77}=V_{4422} \qquad S_{78}=V_{4323} \qquad S_{79}=V_{4423}$$

$$S_{10,1}=V_{4141} \qquad S_{10,2}=2V_{4241} \qquad S_{10,3}=2V_{4341} \qquad S_{10,4}=4T_{4441} \qquad S_{10,5}=V_{4242}$$

$$S_{10,6}=2V_{4342} \qquad S_{10,7}=2V_{4442} \qquad S_{10,8}=V_{4343} \qquad S_{10,9}=4T_{4443} \qquad S_{10,10}=2T_{4444}$$

$$V_{2121}=(\pi\gamma^2/2)[K_\eta^2\Phi_{\eta\eta}+K_\xi^2\Phi_{\xi\xi}+2K_\eta K_\xi\Phi_{\eta\xi}], \quad V_{2323}=(\pi\gamma^2/2)[m_{\alpha\eta}^2\Phi_{\eta\eta}+m_{\alpha\xi}^2+2m_{\alpha\eta}m_{\alpha\xi}\Phi_{\eta\xi}]$$

$$V_{4121}=3\pi\gamma^2 Q[K_\eta\ell_{r\beta\eta}\Phi_{\eta\eta}+K_\xi\ell_{r\beta\xi}\Phi_{\xi\xi}+(K_\eta\ell_{r\beta\xi}+K_\xi\ell_{r\beta\eta})\Phi_{\eta\xi}], \quad V_{4122}=3\pi\gamma^2 Q[C_\eta\ell_{r\beta\eta}\Phi_{\eta\eta}+C_\xi\ell_{r\beta\xi}\Phi_{\xi\xi}+(C_\eta\ell_{r\beta\xi}+C_\xi\ell_{r\beta\eta})\Phi_{\eta\xi}]$$

$$V_{4123}=-3\pi\gamma^2 Q[m_{\alpha\eta}\ell_{r\beta\eta}\Phi_{\eta\eta}+m_{\alpha\xi}\ell_{r\beta\xi}\Phi_{\xi\xi}+(m_{\alpha\eta}\ell_{r\beta\xi}+m_{\alpha\eta}\ell_{r\beta\eta})\Phi_{\eta\xi}]$$

$$V_{4321}=3\pi\gamma^2 Q[K_\eta K_{\alpha\eta}\Phi_{\eta\eta}+K_\xi K_{\alpha\xi}\Phi_{\xi\xi}+(K_\eta K_{\alpha\xi}+K_\xi K_{\alpha\eta})\Phi_{\eta\xi}], \quad V_{4421}=3\pi\gamma^2 F[K_\eta C_{\alpha\eta}\Phi_{\eta\eta}+K_\xi C_{\alpha\xi}\Phi_{\xi\xi}+(K_\eta C_{\alpha\xi}+K_\xi C_{\alpha\eta})\Phi_{\eta\xi}]$$

$$V_{4322}=3\pi\gamma^2 Q[C_\eta K_{\alpha\eta}\Phi_{\eta\eta}+C_\xi K_{\alpha\xi}\Phi_{\xi\xi}+(C_\eta K_{\alpha\xi}+C_\xi K_{\alpha\eta})\Phi_{\eta\xi}] \quad V_{4422}=3\pi\gamma^2 F[C_\eta C_{\alpha\eta}\Phi_{\eta\eta}+C_\xi C_{\alpha\xi}\Phi_{\xi\xi}+(C_\eta C_{\alpha\xi}+C_\xi C_{\alpha\eta})\Phi_{\eta\xi}]$$

$$V_{4323}=-3\pi\gamma^2 Q[m_{\alpha\eta}K_{\alpha\eta}\Phi_{\eta\eta}+m_{\alpha\eta}K_{\alpha\xi}\Phi_{\xi\xi}+(m_{\alpha\eta}K_{\alpha\xi}+m_{\alpha\xi}K_{\alpha\eta})\Phi_{\eta\xi}]$$

$$V_{4423}=-3\pi\gamma^2 F[m_{\alpha\eta}C_{\alpha\eta}\Phi_{\eta\eta}+m_{\alpha\eta}C_{\alpha\xi}\Phi_{\xi\xi}+(m_{\alpha\eta}C_{\alpha\xi}+m_{\alpha\xi}C_{\alpha\eta})\Phi_{\eta\xi}]$$

$$V_{4141}=18\pi\gamma^2 Q^2[\ell^2_{r\beta\eta}\Phi_{\eta\eta}+\ell^2_{r\beta\xi}\Phi_{\xi\xi}+2\ell_{r\beta\eta}\ell_{r\beta\xi}\Phi_{\eta\xi}], \quad V_{4241}=18\pi\gamma^2 Q^2[\ell_{r\beta\eta}\ell_{r\dot\beta\eta}\Phi_{\eta\eta}+\ell_{r\beta\xi}\ell_{r\dot\beta\xi}\Phi_{\xi\xi}+(\ell_{r\beta\eta}\ell_{r\dot\beta\xi}+\ell_{r\beta\xi}\ell_{r\dot\beta\eta})\Phi_{\eta\xi}]$$

$$V_{4341}=18\pi\gamma^2 Q^2[\ell_{r\beta\eta}K_{\alpha\eta}\Phi_{\eta\eta}+\ell_{r\beta\xi}K_{\alpha\xi}\Phi_{\xi\xi}+(\ell_{r\beta\eta}K_{\alpha\xi}+\ell_{r\beta\xi}K_{\alpha\eta})\Phi_{\eta\xi}], \quad V_{4242}=18\pi\gamma^2 Q^2[\ell^2_{r\dot\beta\eta}\Phi_{\eta\eta}+\ell^2_{r\dot\beta\xi}\Phi_{\xi\xi}+2\ell_{r\dot\beta\eta}\ell_{r\dot\beta\xi}\Phi_{\eta\xi}]$$

$$V_{4342}=18\pi\gamma^2 Q^2[\ell_{r\dot\beta\eta}K_{\alpha\eta}\Phi_{\eta\eta}+\ell_{r\dot\beta\xi}K_{\alpha\xi}\Phi_{\xi\xi}+(\ell_{r\dot\beta\eta}K_{\alpha\xi}+\ell_{r\dot\beta\xi}K_{\alpha\eta})\Phi_{\eta\xi}], \quad V_{4343}=18\pi\gamma^2 Q^2[K^2_{\alpha\eta}\Phi_{\eta\eta}+K^2_{\alpha\xi}\Phi_{\xi\xi}+2K_{\alpha\eta}K_{\alpha\xi}\Phi_{\eta\xi}]$$

$$V_{2321}=-(\pi\gamma^2/2)[m_{\alpha\eta}K_\eta\Phi_{\eta\eta}+m_{\alpha\xi}K_\xi\Phi_{\xi\xi}+(m_{\alpha\eta}K_\xi+m_{\alpha\xi}K_\eta)\Phi_{\eta\xi}]$$

$$V_{4123}=-3\pi\gamma^2 Q[\ell_{r\beta\eta}m_{\alpha\eta}\Phi_{\eta\eta}+\ell_{r\beta\xi}m_{\alpha\xi}\Phi_{\xi\xi}+(\ell_{r\beta\eta}m_{\alpha\xi}+\ell_{r\beta\xi}m_{\alpha\eta})\Phi_{\eta\xi}] \tag{19}$$

Numerical Examples

Before showing the numerical results, it is appropriate to examine the range of values for the spectral densities of the excitations, $\Phi_{\eta\eta}$ and $\Phi_{\xi\xi}$, which may be encountered in practical flight conditions. These spectral densities have been nondimensionalized. For example, $\Phi_{\eta\eta}$ is the spectral density of nondimensional turbulence velocity $\eta(t)$ vs nondimensional frequency ω/Ω. This can be converted into one vs the physical frequency ω(rad/s) as follows:

$$\bar{\Phi}_{\eta\eta}=\Phi_{\eta\eta}/\Omega$$

Since $\eta(t)$ is nondimensionalized with respect to the blade tip speed ΩR (m/s), further conversion to the spectral density of the physical turbulence velocity yields

$$\bar{\bar{\Phi}}_{VV}=\Omega R^2\Phi_{\eta\eta}(\text{m/s})^2/\text{rad/s}$$

Perhaps the most basic statistical information about a turbulence field is its mean square value, which is equal to the area between the $\bar{\bar{\Phi}}_{VV}(\omega)$ curve and ω axis. Such a curve is generally quite smooth, and it can often be idealized as being constant over some frequency range of practical interest. To be more specific, we may assume that $\bar{\bar{\Phi}}_{VV}(\omega)$ is constant over $-\omega_c<\omega<\omega_c$ and is zero outside this range. This ideal model is called truncated white noise, and ω_c is called the cutoff frequency.

In our analysis the turbulence is untruncated white noise for which the spectral density is the same constant over the entire frequency range. Thus the mean-square value of untruncated white noise is, of course, unbounded. However, it is well known that the response of a dynamic system to a truncated white noise is approximately the same as that to an untruncated one, provided that the major frequency contents in the response lie within the truncation interval $[-\omega_c,\omega_c]$ of the excitation.

In practical applications, we measure the mean square value σ_V^2 and the spectral density of the excitation $\bar{\bar{\Phi}}_{VV}$. If the spectral level varies smoothly within a certain frequency range through which most energy of excitation is supplied to the response of the dynamic system, then substitution of the real excitation process by truncated white noise is permissible.

The cutoff frequency ω_c is related to the mean square value of the excitation σ_V^2 by

$$\sigma_V^2=2\omega_c\bar{\bar{\Phi}}_{VV}$$

where $\bar{\bar{\Phi}}_{VV}$ is the idealized constant spectral level of the truncated white noise which, of course, must be comparable to the measured spectral level within the frequency range of practical interest.

In the first column of Table 1 the values for the nondimensional spectra $\Phi_{\eta\eta}$ used in the numerical computations are listed. The second column gives the corresponding physical spectrum, $\bar{\bar{\Phi}}_{VV}$, and the other three columns show the corresponding root mean square (rms) values of the excitation for different truncation frequencies in terms of the ratio ω_c/Ω. All are computed on the basis of $\Omega=7\pi$ and $R=5$ m. The values should be adjusted for different rotation speeds and blade radii.

It is known from various measurements that a typical atmospheric turbulence of geothermal origin has an rms value of 3 m/s or less. However, wakes from adjacent blades of the same rotor or another rotor can create much higher velocity fluctuations in the airflow field. Thus, if both sources of turbulence are taken into consideration, the overall spectral level may reach a normalized value $\Phi_{\eta\eta}=0.003$. This intensity may be considered to be near the upper limit under practical circumstances.

In obtaining numerical results to be presented herein, higher values of the turbulence spectral level have been used sometimes so that the change from the nonturbulence condition can be seen more clearly. When a higher-than-usual turbulence level is used in the computation, the change in the stability boundary is exaggerated; however, the same general tendency is expected to occur at realistic turbulence spectral levels.

Figure 1 shows the stability boundaries plotted on the γ vs p^2 plane for the first and second stochastic moments of the uncoupled flapping response at an advance ratio $\mu=2.4$, and corresponding to different turbulence levels. The two turbulence velocity components, parallel and perpendicular to the flight path, respectively, are assumed to be uncorrelated but having the same spectra. The stability region is seen to be reduced by turbulence; the higher the turbulence level, the smaller the stable region. As expected, the second-moment stability region is always included in the first-moment stability region. The same conclusion can be reached using the Schwartz inequality. Also included in the figure is the nonturbulence case as a baseline for comparison. This baseline agrees with the one previously obtained by Sissingh[5] using an

Table 1 Corresponding normalized spectrum level, real spectrum level and rms of excitation $V(t)$

$\Phi_{\eta\eta}$	$\bar{\bar{\Phi}}_{VV}$, $\frac{(\text{m/s})^2}{\text{rad/s}}$	σ_V, m/s: $\omega_c/\Omega=1$	$\omega_c/\Omega=2$	$\omega_c/\Omega=3$
3.18×10^{-4}	0.175	2.77	3.92	4.80
6.36×10^{-4}	0.350	3.92	5.54	6.70
9.55×10^{-4}	0.525	4.80	6.79	8.31
1.59×10^{-3}	0.875	6.19	8.77	10.73
2.23×10^{-3}	1.225	7.33	10.37	12.70
3.18×10^{-3}	1.75	8.76	12.40	15.18

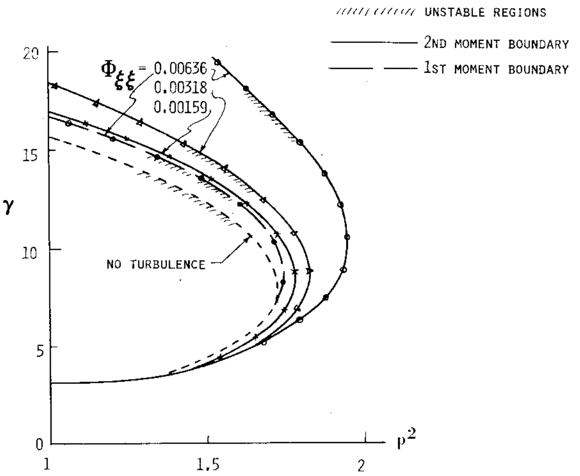

Fig. 1 First- and second-moment stability boundaries for uncoupled flapping motion: $\mu = 2.4$; $\Phi_{\xi\xi} = \Phi_{\eta\eta}$, $\Phi_{\xi\eta} = 0$.

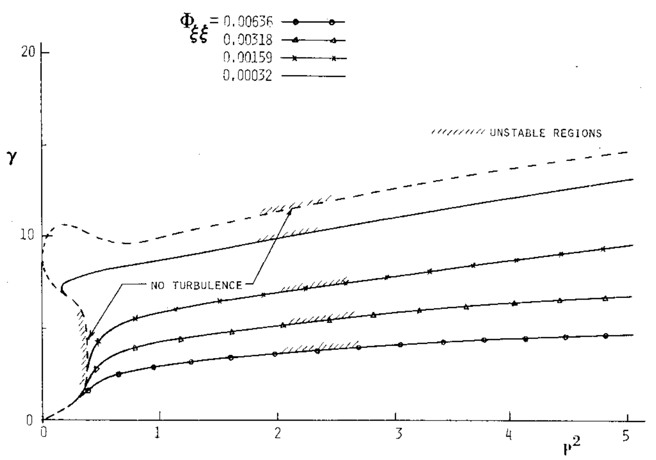

Fig. 2 Second-moment stability boundaries for coupled flapping-torsional motion: $\mu = 1.6$; $\omega_\alpha = 10$; $F = 0.24$; $Q = 15$; $\Phi_{\xi\xi} = \Phi_{\eta\eta}, \Phi_{\xi\eta} = 0$.

analog computer, and verified later by Peters and Hohenemser[6] using the numerical Floquet matrix method. Since vector $\{X_1, X_2\} = \{\beta, \dot{\beta}\}$ is treated as the response, the asymptotic stabiliity for the first moment only assures that $E[\beta]$ and $E[\dot{\beta}]$ approach zero. This condition is not as useful as the one obtained previously[2] for the amplitude $A = (\beta^2 + p^{-2}\dot{\beta}^2)$ in the case of hovering flights. Under normal conditions, the first-moment stability boundaries for the uncoupled flapping motion do not deviate much from the baseline. The one that is shown in Fig. 1 corresponds to an unusually high turbulence level of 0.00636 in order to demonstrate the general nature of such curves. The more useful stability boundaries are the ones for the second stochastic moments shown here for three spectral levels of 0.00159, 0.00318, and 0.00636.

The advance ratio $\mu = 2.4$ was used in previous deterministic analyses,[1,6] although it is much too high to be expected of present helicopters and, perhaps, also high for possible future designs. The present stochastic analysis has shown that the flapping motion remains extremely stable in turbulent flows.

Figure 2 shows the second-moment stability boundaries for the coupled flapping torsional motion, again when the longitudinal and lateral turbulence velocities have the same spectral level but are uncorrelated. The response vector is now $\{X_1, X_2, X_3, X_4\} = \{\beta, \dot{\beta}, \alpha, \dot{\alpha}\}$, combining into ten second-moments $E[X_1^2]$, $E[X_1X_2]$, $E[X_1X_3]$, $E[X_1X_4]$, $E[X_2^2]$, $E[X_2X_3]$, $E[X_2X_4]$, $E[X_3^2]$, $E[X_3X_4]$, and $E[X_4^2]$. The stability boundaries for the first moments are not shown in the figure for lack of practical importance. It is seen that the stability boundary of the coupled motion deviates substantially from the baseline of the nonturbulence case, due to a rather low-turbulence spectral level of $\Phi_{\xi\xi} = 0.000318$, which can occur even from a natural geothermal source. All the stability boundaries shown in Fig. 2, including the baseline, are nearly straight on the γ-p^2 plane in the range $p^2 > 1$, while rapid change takes place in the lower p^2 range. It is well known from the deterministic theory of parametric excitation involving the Mathieu-Hill-type equations that a primary instability occurs at $p^2 = 0.25$. This accounts for the departure of stability boundaries in the region $0 < p^2 < 1$ from the general trend appearing in the region $p^2 > 1$. In helicopter dynamics, the flapping stiffness parameter p^2 of the blade is always greater than one. Therefore, results for $0 < p^2 < 1$ have no practical significance but are included here for completeness.

Concluding Remarks

The most crucial assumption in the analysis is the relative shortness of the correlation time of the turbulence field when viewed from a coordinate system attached to the blade, as compared with the relaxation time of the dynamic system. This assumption permits the use of a Markov model and the Itô calculus in the mathematical theory of probability. This assumption is reasonable if the turbulence is carried by a high convection velocity relative to the blade. If the correlation time of the excitation field is not sufficiently short, then one remedy is to model the excitations $\xi(t)$ and $\eta(t)$ as outputs of suitable linear filters which are fed with white noise inputs. The number of dimensions of the resulting Markov vector will be increased to include the dimensions of the linear filters; hence, the additional algebraic and computational complexities. However, the underlying principle remains unchanged.

The use of the Wang and Akai conversion rule, which is much simpler than the more general stochastic averaging procedure of Stratonovich and Khasminiskii, requires the white noise assumption for the initial inputs. Since the correlation time of a white noise process is infinitely short, a comparison with the length of the system relaxation time is not necessary. The implication is that the analysis is valid regardless of the magnitude of damping in the system. This may be important since the aerodynamic damping of blade torsion varies greatly in forward flight.

The main conclusions reached in this exploratory study are summarized as follows:

1) The flapping and torsional motions of a rotor blade operating in a three-dimensional turbulence field are governed by differential equations with periodic and random coefficients.

2) For the uncoupled flapping motion or the coupled flapping torsional motion, the two horizontal turbulence components affect the system stability, but the vertical turbulence component contributes only to the nonparametric external force.

3) The present analysis is valid for an arbitrary advance ratio.

4) Numerical calculations have confirmed the fact that the second-moment stability guarantees the first-moment stability.

5) The second-moment stability boundaries of the uncoupled flapping motion deviate only slightly from the nonturbulence baseline, even under high-level turbulence excitations. Thus, it remains very stable under normal circumstances.

6) Contrary to the case of uncoupled flapping, the second-moment stability boundaries of the coupled flapping torsional motion under realistic turbulence levels may differ significantly from the nonturbulence baseline.

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